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An infinite product formula for $U_q(sl(2))$ dynamical coboundary element

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Abstract

We give a short summary of results and conjectures in the theory of dynamical quantum group related to the dynamical coboundary equation also known as IRF-vertex transform. Babelon has shown that the dynamical twist $F(x)$ of $U_q(sl(2))$ is a dynamical coboundary $M(x)$, i.e. $F(x)M_1(xq^{h_2})M_2(x) = \Delta(M(x))$. We give a new formula for this element $M(x)$ as an infinite product and give a new proof of the coboundary relation. Our proof involves the quantum Weyl group element, giving a possible hint for the generalization to higher rank case.

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1. Introduction

While participating in the conference 'Recent Advances in the Theory of Quantum Integrable Systems 2003', LAPTH, Annecy-Le-Vieux, we have seen that there was some interest in very preliminary work we had done on the understanding of the shifted coboundary element some years ago. We have therefore decided to publish these results in the proceedings of the conference.

The theory of dynamical quantum groups is nowadays a well-established part of mathematics, see the review by Etingof [1]. This theory originated from the notion of the dynamical Yang–Baxter equation, which arose in the work of Gervais–Neveu on Liouville theory [2] and was formalized first by Felder [3] who also understood its relation to IRF statistical models.

In its work on quantum Liouville on a lattice [4], Babelon was the first to understand the universal aspects of the dynamical Yang–Baxter equation. He introduced the notion of dynamical twist $F(x) \in U_q(sl(2))^{\otimes 2}$ and gave an exact formula for $F(x)$. Quite remarkably

[1] Work supported by CNRS.

he obtained the explicit formula of $F(x)$ by noting that $F(x)$ is a dynamical coboundary, i.e there exists an explicit invertible element $M(x) \in U_q(sl(2))$ such that

$$F(x) = \Delta(M(x))M_2(x)^{-1}(M_1(xq^{h_2}))^{-1}. \qquad (1)$$

Twelve years after this work, the theory of dynamical quantum groups is now well understood for $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is a Kac–Moody algebra of affine or finite type. An important result is the so-called 'linear equation' [5–7] which allows us to express the dynamical twist as an infinite product of 'dressed' R-matrices. It is quite surprising that the theory of dynamical quantum groups has made no significant progress in the understanding of the coboundary element. See however the articles [8–10].

We will now recall the present situation as well as conjectures. We will give in the next section a new formula for $M(x)$ expressed as an infinite product. Using this formula we will show that we can recover (after some work!) the infinite product formula for $F(x)$. The present work stems from an unsuccessful attempt to understand the structure of the coboundary for higher rank case. However we hope that our work will trigger some further study of the coboundary element and its relation to the quantum Weyl group.

We first recall the results of Babelon using his notation. In the following we will denote $[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}$ as well as $(z)_q = q^{z-1}[z]_q$. The q-exponential is defined as

$$\exp_q(z) = \sum_{n=0}^{+\infty} \frac{z^n}{(n)_q!} \qquad (2)$$

where $(n)_q! = (n)_q \cdots (1)_q$.

It satisfies

$$\exp_q(z)\exp_{q^{-1}}(-z) = 1 \qquad (3)$$

and if $xy = q^2yx$

$$\exp_q(x + y) = \exp_q(y)\exp_q(x) \qquad (4)$$

$$\exp_q(x)\exp_q(y) = \exp_q(y)\exp_q((1 - q^{-2})xy)\exp_q(y). \qquad (5)$$

Let us define $U_q(sl(2))$ to be the Hopf algebra generated by E_+, E_-, h satisfying

$$[h, E_\pm] = \pm 2E_\pm \qquad [E_+, E_-] = [h]_q \qquad (6)$$

$$\Delta(E_\pm) = E_\pm \otimes q^{h/2} + q^{-h/2} \otimes E_\pm. \qquad (7)$$

We will define $e = q^{h/2}E_+$, $f = E_-q^{-h/2}$.

We can express the Drinfeld universal R matrix as

$$R = q^{\frac{h\otimes h}{2}}\hat{R} \qquad (8)$$

with $\hat{R} = \exp_{q^{-1}}((q - q^{-1})e \otimes f)$.

The dynamical twist F is a function $F : \mathbb{C}^* \to U_q(sl(2))^{\otimes 2}$ of zero weight, i.e. $[F(x), \Delta(h)] = 0$, satisfying the dynamical cocycle equation

$$F_{12,3}(x)F_{12}(xq^{h_3}) = F_{1,23}(x)F_{23}(x) \qquad (9)$$

where we have denoted $F_{12,3}(x) = (\Delta \otimes id)(F(x))$.

An explicit solution was constructed by Babelon and reads

$$F(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k(q - q^{-1})^k}{(k)_{q^{-1}}!} \frac{q^{k(1-k)}}{\prod_{p=1}^{k}(1 - x^{-2}q^{-2p-2h_2})} \mathrm{e}^k \otimes f^k. \qquad (10)$$

If we define $R(x) = F_{21}(x)^{-1} R F_{12}(x)$, $R(x)$ is a solution of the dynamical Yang–Baxter equation, i.e.

$$R_{12}(x) R_{13}(xq^{h_2}) R_{23}(x) = R_{23}(xq^{h_1}) R_{13}(x) R_{12}(xq^{h_3}). \tag{11}$$

It was shown in [5] that $F(x)$ satisfies the linear equation

$$F(x) B_2(x) = \hat{R}^{-1} B_2(x) F(x) \tag{12}$$

where $B(x) = x^h q^{h^2/2}$.

This equation can be iterated and we obtain an expression of $F(x)$ in terms of an infinite product [6]:

$$F(x) = \prod_{k=0}^{+\infty} (B_2(x)^k \hat{R}^{-1} B_2(x)^{-k}) \tag{13}$$

$$= \prod_{k=0}^{+\infty} \exp_q(-(q - q^{-1}) x^{-2k} q^{-2k(1+h_2)} e \otimes f). \tag{14}$$

This product is shown to be convergent in finite-dimensional representations and for x sufficiently large. We can also express $F(x)$ as

$$F(x) = \prod_{k=0}^{+\infty} (B_2(x)^{-k-1} \hat{R} B_2(x)^{k+1}) \tag{15}$$

which is convergent in finite-dimensional representations and for x sufficiently small.

Babelon has shown [4] that if we define $M : \mathbb{C}^* \to U_q(sl(2))$ by

$$M(x) = \sum_{n,m=0}^{+\infty} \frac{(-1)^m x^m q^{n(n-1)/2+m(n-m)}}{[m]_q! [n]_q! \prod_{p=1}^{n} (xq^p - x^{-1} q^{-p})} E_+^n E_-^m q^{(n+m)h/2} \tag{16}$$

we have

$$F(x) M_1(xq^{h_2}) M_2(x) = \Delta(M(x)). \tag{17}$$

Note that $M(x)$ is not of zero weight, i.e. $[M(x), h] \neq 0$. From this relation we obtain that

$$R(x) M_1(xq^{h_2}) M_2(x) = M_2(xq^{h_1}) M_1(x) R \tag{18}$$

i.e. one can absorb all the dynamic through the use of $M(x)$.

Note that $M(x)$ relates the dynamical R-matrix $R(x)$, which in the affine case would give the IRF Boltzmann weights, to the R-matrix, which in the affine case would give the vertex Boltzmann weights. This is the reason why $M(x)$ is also called by extension the IRF–vertex transform. Such a type of M, in matrix form, has been computed by Baxter in [14] and relates the IRF model to the eight-vertex solution.

All these results have been obtained in the $U_q(sl(2))$ case. Since we now have a good theory of dynamical quantum groups in the $U_q(\mathfrak{g})$ case, one can expect that such results can be generalized to the higher rank case.

We will use the notation of [6]. We denote $F(x)$ the dynamical twist in the $U_q(\mathfrak{g})$ case where $x = q^\lambda$ with $\lambda \in \mathfrak{h}$. The linear equation still applies in this case and we have

$$F(x) = \prod_{k=0}^{+\infty} (B_2(x)^k \hat{R}^{-1} B_2(x)^{-k}) \tag{19}$$

where $B(x)$ and $\hat{R}$ are the straightforward generalization to the higher rank as explained in [6].

The product is convergent in finite-dimensional representations and for λ sufficiently large.

The first result on the coboundary element for the higher rank case was obtained by Cremmer–Gervais [11] in the $sl(3)$ case and by Bilal–Gervais [12] in the $sl(n+1)$ case, where they have shown that, in the fundamental representation of $sl(n+1)$, one can absorb all the dynamics of $R(x)$ and one obtains

$$R(x)M_1(xq^{h_2})M_2(x) = M_2(xq^{h_1})M_1(x)R_{CG} \tag{20}$$

with $M(x)$ is an $(n+1)\times(n+1)$ explicit matrix (of VanderMonde type) and R_{CG} satisfies the Yang–Baxter equation.

Quite surprisingly, R_{CG} is not the standard solution of the Yang–Baxter equation and is not of zero weight. Its classical expansion $R_{CG} = 1 + \hbar r_{CG} + o(\hbar)$ is associated with a particular solution of Belavin–Drinfeld classification. In the $sl(n+1)$ case it corresponds to taking the following Belavin–Drinfeld triple:

$$\begin{aligned}
&\Gamma = \{\alpha_1, \ldots, \alpha_n\}\\
&\Gamma_1 = \{\alpha_1, \ldots, \alpha_{n-1}\} \qquad \Gamma_2 = \{\alpha_2, \ldots, \alpha_n\}\\
&T : \Gamma_1 \to \Gamma_2 \quad \alpha_i \mapsto \alpha_{i+1}.
\end{aligned}$$

Unfortunately, all these computations have been done in the fundamental representation and give no real hints towards a precise universal formulation. The subject has seen a dramatic advance with the result of Etingof *et al* [13], where they have shown how to quantize explicitly all classical solutions of the Yang–Baxter equation given by a Belavin–Drinfeld triple. They have constructed for all Belavin–Drinfeld triple T a twist $J_T = J \in U_q(\mathfrak{g})^{\otimes 2}$, which satisfies

$$J_{12,3}J_{12} = J_{1,23}J_{23} \tag{21}$$

and is such that $J_{21}^{-1}RJ_{12}$ satisfies the Yang–Baxter equation and is a quantization of the classical r_T matrix associated with T. The expression of J was obtained through a nice use of dynamical quantum groups and of a modification of the linear equation. Finally J is expressed as a finite product of explicit invertible elements. Therefore this result provides the answer to the construction of explicit universal formulae for the non-standard solution of a Yang–Baxter equation such as the Cremmer–Gervais one.

It is therefore tempting to formulate the following conjectures:

Conjecture 1. When $\mathfrak{g} = sl(n+1)$, there exists $M : \mathbb{C}^{*n} \to U_q(\mathfrak{g})$ such that

$$F(x)M_1(xq^{h_2})M_2(x) = \Delta(M(x))J_T \tag{22}$$

with T given by the previous Belavin–Drinfeld triple.

Conjecture 2. A solution to this equation is given in terms of an infinite product of explicit invertible elements of $U_q(\mathfrak{g})$.

In the following we will reconsider these conjectures in the $U_q(sl(2))$ case. Conjecture 1, in this case, is already solved, but we will give a new formula for $M(x)$ in this case and show that conjecture 2 is satisfied. We will then check directly on the infinite product expression that the coboundary equation is satisfied.

2. The infinite product expression

Let $Q : \mathbb{C}^* \to U_q(sl(2))$. We denote δQ the function $\delta Q(x) = \Delta(Q(x))Q_2(x)^{-1}(Q_1(xq^{h_2}))^{-1}$. A dynamical group like element is a function $g : \mathbb{C}^* \to U_q(sl(2))$ such that $\delta g(x) = 1$. One can easily construct such dynamical group like elements. Apart from the constant group like element, one can also define, for example,

$$\text{Example 1} \quad g(x) = B(x) \tag{23}$$

$$\text{Example 2} \quad g(x) = (x;q)_h = \frac{(x;q)_\infty}{(xq^h;q)_\infty}. \tag{24}$$

If $g : \mathbb{C}^* \to U_q(h)$ is a dynamical group like element, then

$$\delta(gQ)(x) = \Delta(g(x))\delta(Q(x))(\Delta(g(x)))^{-1}.$$

As a result if Q satisfies the coboundary equation (1), from the h invariance of $F(x)$, we obtain that gQ is also a solution of the coboundary equation.

We will now show that there exists a solution of the coboundary equation which is written as a simple infinite product formula. We think that this proof, although a bit intricate, could be generalized to the higher rank case in spite of our unsuccessful attempt.

Proposition 1. *Let $N : \mathbb{C}^* \to U_q(sl(2))$ be defined as*

$$N(x) = N_-(x)N_+(x) \tag{25}$$

with

$$N_+(x) = \exp_{q^{-1}}(-xe) \tag{26}$$

$$N_-(x) = \prod_{k=+\infty}^{0} \exp_{q^{-1}}(q^{-(2k+1)(h+1)}x^{-2k-1}f). \tag{27}$$

$N(x)$ satisfies the coboundary equation.

In order to prove this proposition, we first begin by proving a lemma interesting in itself. We use the notation of the article [6]: let $\mathfrak{g}$ be a finite simple Lie algebra of rank r and denote by $F : \mathbb{C}^{*r} \to U_q(\mathfrak{g})^{\otimes 2}$ the solution of the dynamical cocycle equation.

Lemma 1. *Let u, v be maps from $\mathbb{C}^{*r}$ to $U_q(\mathfrak{g})$ and $\mathcal{J}$ an invertible element of $U_q(\mathfrak{g})^{\otimes 2}$.*

We define

$$V^{(p)}(x) = \prod_{k=p}^{1}(B(x)^k v(x)B(x)^{-k}) \qquad V^{(0)}(x) = 1 \qquad M^{(0)}(x) = u(x)^{-1}$$

$$M^{(p)}(x) = V^{(p)}(x)M^{(0)}(x) \qquad F^{(p)}(x) = \Delta(M^{(p)}(x))\mathcal{J}M^{(p)}(x)_2^{-1}M^{(p)}(xq^{h_2})_1^{-1}$$

and we define $G(x) \equiv F^{(0)}(x)^{-1}\Delta(B(x))^{-1}F^{(1)}(x)\Delta(B(x))$.

If the following relation is satisfied:

$$\left[G(x)v(xq^{h_2})_1B(xq^{h_2})_1^{-1}, M^{(0)}(xq^{h_2})_1v(x)_2B(x)_2^{-1}M^{(0)}(xq^{h_2})_1^{-1}\right] = 0 \tag{28}$$

as well as the asymptotic relations

$$\lim_{k\to+\infty} B(x)^k v(x)^{-1}B(x)^{-k} = 1$$

$$\lim_{k\to+\infty} \Delta(B(x))^k F^{(0)}(x)\Delta(B(x))^{-k} = 1$$

$$\lim_{k\to+\infty} \Delta(B(x))^k G(x)\Delta(B(x))^{-k} = \hat{R}^{-1}$$

then

$$\lim_{k\to+\infty} F^{(k)}(x) = F(x).$$

Proof. We first remark the following recursion relations:

$$V^{(p)}(x) = B(x)V^{(p-1)}(x)v(x)B(x)^{-1}$$
$$M^{(p)}(x) = B(x)M^{(p-1)}(x)(M^{(0)}(x)^{-1}v(x)B(x)^{-1}M^{(0)}(x)).$$

This allows us to deduce the following recursion relation on $F^{(p)}(x)$:

$$\begin{aligned}
F^{(p)}(x) &= \Delta(B(x))\Delta(M^{(p-1)}(x))\Delta(M^{(0)}(x)^{-1}v(x)B(x)^{-1}M^{(0)}(x))\,\mathcal{J}M^{(p)}(x)_2^{-1} \\
&\quad\times\ M^{(p)}(xq^{h_2})_1^{-1} \\
&= \Delta(B(x))F^{(p-1)}(x)\big(M^{(p-1)}(xq^{h_2})_1M^{(p-1)}(x)_2\,\mathcal{J}^{-1}\Delta(M^{(0)}(x)^{-1}v(x)B(x)^{-1} \\
&\quad\times\ M^{(0)}(x))\mathcal{J}M^{(0)}(x)_2^{-1}V^{(p)}(x)_2^{-1}\,M^{(0)}(xq^{h_2})_1^{-1}V^{(p)}(xq^{h_2})_1^{-1}\big) \\
&= \Delta(B(x))F^{(p-1)}(x)\,\big(M^{(p-1)}(xq^{h_2})_1M^{(p-1)}(x)_2\,M^{(0)}(x)_2^{-1}M^{(0)}(xq^{h_2})_1^{-1} \\
&\quad\times\ G(x)v(xq^{h_2})_1B(xq^{h_2})_1^{-1}M^{(0)}(xq^{h_2})_1V^{(p-1)}(x)_2^{-1}B(x)_2^{-1}M^{(0)}(xq^{h_2})_1^{-1} \\
&\quad\times\ B(xq^{h_2})_1v(xq^{h_2})_1^{-1}V^{(p-1)}(xq^{h_2})_1^{-1}B(xq^{h_2})_1^{-1}\big).
\end{aligned}$$

In order to check the last equality, it is sufficient to show the following equality after having expanded $V^{(p)}(x)$ in terms of $V^{(p-1)}(x)$:

$$\begin{aligned}
&\mathcal{J}^{-1}\Delta(M^{(0)}(x)^{-1}v(x)B(x)^{-1}M^{(0)}(x))\mathcal{J}M^{(0)}(x)_2^{-1}V^{(p)}(x)_2^{-1} \\
&\quad= M^{(0)}(x)_2^{-1}M^{(0)}(xq^{h_2})_1^{-1}G(x)v(xq^{h_2})_1B(xq^{h_2})_1^{-1}M^{(0)}(xq^{h_2})_1V^{(p-1)}(x)_2^{-1}B(x)_2^{-1}.
\end{aligned}$$

The left-hand side of this equation is equal to

$$\begin{aligned}
&M^{(0)}(x)_2^{-1}M^{(0)}(xq^{h_2})_1^{-1}F^{(0)}(x)^{-1}\Delta(B(x)^{-1})F^{(1)}(x)\Delta(B(x)) \\
&\qquad\times\Delta(B(x)^{-1})M^{(1)}(xq^{h_2})_1M^{(1)}(x)_2M^{(0)}(x)_2^{-1}V^{(p)}(x)_2^{-1}
\end{aligned}$$

which is shown to be equal to the right-hand side of this equation using the definition of $G(x)$ and the equality,

$$\begin{aligned}
&\Delta(B(x)^{-1})M^{(1)}(xq^{h_2})_1M^{(1)}(x)_2M^{(0)}(x)_2^{-1}V^{(p)}(x)_2^{-1} \\
&\qquad= v(xq^{h_2})_1B(xq^{h_2})_1^{-1}M^{(0)}(xq^{h_2})_1V^{(p-1)}(x)_2^{-1}B(x)_2^{-1}.
\end{aligned}$$

If $G(x)$ satisfies the commutation relation of the hypothesis it necessarily satisfies

$$\big[G(x)v(xq^{h_2})_1B(xq^{h_2})_1^{-1},\, M^{(0)}(xq^{h_2})_1B(x)_2v(x)_2^{-1}M^{(0)}(xq^{h_2})_1^{-1}\big] = 0 \qquad (29)$$

then, by recursion, we deduce

$$\big[G(x)v(xq^{h_2})_1B(xq^{h_2})_1^{-1},\, M^{(0)}(xq^{h_2})_1V^{(p-1)}(x)_2^{-1}B(x)_2^{p-1}M^{(0)}(xq^{h_2})_1^{-1}\big] = 0.$$

We will now show that this relation implies

$$\begin{aligned}
&\big(M^{(p-1)}(xq^{h_2})_1M^{(p-1)}(x)_2M^{(0)}(x)_2^{-1}M^{(0)}(xq^{h_2})_1^{-1}G(x)v(xq^{h_2})_1B(xq^{h_2})_1^{-1} \\
&\qquad\times M^{(0)}(xq^{h_2})_1V^{(p-1)}(x)_2^{-1}B(x)_2^{-1}M^{(0)}(xq^{h_2})_1^{-1}B(xq^{h_2})_1v(xq^{h_2})_1^{-1} \\
&\qquad\times V^{(p-1)}(xq^{h_2})_1^{-1}B(xq^{h_2})_1^{-1} \\
&\quad= V^{(p-1)}(xq^{h_2})_1B(x)_2^{p-1}G(x)\big(V^{(p-1)}(xq^{h_2})_1B(x)_2^{p-1}\big)^{-1}\Delta(B(x))^{-1}.
\end{aligned}$$

Indeed in this last equation, by eliminating $V^{(p-1)}(xq^{h_2})_1$ on the left and eliminating $\big(V^{(p-1)}(xq^{h_2})_1B(x)_2^{p-1}\big)^{-1}$ on the right, we obtain the previous relation.

It is now easy to simplify the recursion relation for $F^{(p)}(x)$:

$$\begin{aligned}
F^{(p)}(x) &= \Delta(B(x))F^{(p-1)}(x)\big(V^{(p-1)}(xq^{h_2})_1B(x)_2^{p-1}\big) \\
&\qquad\times G(x)\big(V^{(p-1)}(xq^{h_2})_1B(x)_2^{p-1}\big)^{-1}\Delta(B(x))^{-1}.
\end{aligned}$$

Using the previous relation, as well as the formula

$$B(x)_2^{-i}V^{(i-1)}(xq^{h_2})_1^{-1}\Delta\,(B(x))^{-1}\,V^{(i)}(xq^{h_2})_1B(x)_2^{i+1} = v(xq^{h_2})_1B(xq^{h_2})_1^{-1}$$

we obtain finally

$$F^{(p)}(x) = (\Delta(B(x))^p F^{(0)}(x)\Delta(B(x))^{-p}) \prod_{i=0}^{p-1} \big(B(x)_2^i \big(\Delta(B(x))^{p-i} G(x)\Delta(B(x))^{-(p-i)}\big)$$
$$\times \big(B(xq^{h_2})_1^{p-i} v(xq^{h_2})_1 B(xq^{h_2})_1^{-(p-i)}\big) B(x)_2^{-i}\big)$$
$$\times \prod_{i=1}^{p} \big(B(xq^{h_2})_1^i v(xq^{h_2})_1^{-1} B(xq^{h_2})_1^{-i}\big).$$

Using the asymptotic properties, we have

$$\lim_{p\to+\infty} \big(\Delta(B(x))^p F^{(0)}(x)\Delta\,(B(x))^{-p}\big) = 1$$

$$\lim_{p\to+\infty} \prod_{i=0}^{p/2-1} \big(B(x)_2^i \big(\Delta(B(x))^{p-i} G(x)\Delta(B(x))^{-(p-i)}\big)$$
$$\times \big(B(xq^{h_2})_1^{p-i} v(xq^{h_2})_1 B(xq^{h_2})_1^{-(p-i)}\big) B(x)_2^{-i}\big) = F(x)$$

$$\lim_{p\to+\infty} \prod_{i=p/2}^{p-1} \big(B(x)_2^i \big(\Delta(B(x))^{p-i} G(x)\Delta(B(x))^{-(p-i)}\big)$$
$$\times \big(B(xq^{h_2})_1^{p-i} v(xq^{h_2})_1 B(xq^{h_2})_1^{-(p-i)}\big) B(x)_2^{-i}\big)$$
$$\times \prod_{i=1}^{p/2} \big(B(xq^{h_2})_1^i v(xq^{h_2})_1^{-1} B(xq^{h_2})_1^{-i}\big) = 1$$

$$\lim_{p\to+\infty} \prod_{i=p/2+1}^{p} \big(B(xq^{h_2})_1^i v(xq^{h_2})_1^{-1} B(xq^{h_2})_1^{-i}\big) = 1$$

we obtain the announced result, i.e.

$$\lim_{p\to+\infty} F^{(p)}(x) = F(x).$$

□

We now prove the proposition using this lemma.

Proof. We restrict to $U_q(sl(2))$ and define the following elements:

$$u = \exp_q(e) \qquad v = \exp_{q^{-1}}(f) \qquad b(x) = q^{\frac{h^2}{4}} x^{\frac{h}{2}}. \tag{30}$$

We set $v(x) = b(x)^{-1} v b(x), u(x) = x^{h/2} u x^{-h/2}$, and set $\mathcal{J} = 1$. They satisfy the following relations:

$$\Delta(u) = u_2 q^{\frac{h_1 h_2}{2}} u_1 q^{-\frac{h_1 h_2}{2}} \tag{31}$$

$$\Delta(v) = q^{\frac{h_1 h_2}{2}} v_2 q^{-\frac{h_1 h_2}{2}} v_1 \tag{32}$$

$$u_1 q^{\frac{h_1^2}{4}} R_{12}^{-1} q^{-\frac{h_1^2}{4}} v_2^{-1} = q^{-\frac{h_1 h_2}{2}} v_2^{-1} q^{\frac{h_1 h_2}{2}} u_1 q^{-\frac{h_1 h_2}{2}}. \tag{33}$$

Using these relations we verify that $F^{(0)}(x) = 1$, $F^{(1)}(x) = \hat{R}^{-1} = G(x)$. Relation (29) is shown to be satisfied using (32) and (33).

It remains to show the asymptotic properties. The first one is satisfied in finite-dimensional representations and for x sufficiently large, the second and the third ones are trivial because the sequences are constant. □

We have designed the proof of the coboundary equation having in mind to generalize it to the higher rank case. The important relation (29) was a direct consequence of the two relations (32), (33). We can rewrite these relations using the quantum Weyl group element. Indeed let w be defined by

$$w = vq^{-\frac{h^2}{4}}u^{-1}q^{-\frac{h^2}{4}}v = u^{-1}q^{-\frac{h^2}{4}}vq^{-\frac{h^2}{4}}u^{-1}. \tag{34}$$

This element satisfies

$$whw^{-1} = -h \qquad wew^{-1} = -q^{-h-1}f \qquad wfw^{-1} = -eq^{h+1} \tag{35}$$

$$\Delta(w) = \hat{R}^{-1}w_1w_2. \tag{36}$$

We remark that the pair (u, v) solution of (31)–(33) can be constructed solely from u and w. Precisely, if u and w satisfy the relations (31), (35), (36) as well as the relation

$$wuw = u^{-1}q^{-\frac{h^2}{2}}wu^{-1} \tag{37}$$

then we can define $v = q^{-\frac{h^2}{4}}wu^{-1}w^{-1}q^{\frac{h^2}{4}}$ and the couple (u, v) satisfies the relations (31)–(33).

This is why we strongly suspect that the structure of the proof will remain unchanged in the $U_q(sl(n+1))$: w will be the longest element in the quantum Weyl group and the only unknown element to us is the element u, which definition must generalize the relations (31), (37) by incorporating J_T.

We now come back to the relation between this infinite production solution and the solution of Babelon: they are related by a dynamical group like element. It amounts to reordering the e and f in Babelon's formula and to transforming a sum in a product. We found it in a very indirect way using the relation between the matrix element of $M(x)$ and the $3j$-symbols of $U_q(sl(2))$ [8]. It was very surprising for us that such a factorization occurs because the formula for $M(x)$ involves the factor q^{mn} which prevents to disentangle the double sum in the opposite factorization. They are related by a dynamical group like element:

Proposition 2. *Let $M(x)$ be the element defined by (16), it satisfies $N(x) = (q^2x^2; q^2)_h M(x)$.*

Proof. We first reorder the e and the f in Babelon's formula. We will first show that

$$M(x) = \tilde{N}_-(x)N_+(x) \tag{38}$$

where

$$\tilde{N}_-(x) = \sum_{k=0}^{+\infty} \frac{q^{\frac{k(k+1)}{2}}(-z)^k q^{kh}}{(q^2z^2; q^2)_{h+k}[k]!} f^k. \tag{39}$$

By projecting on each weight space for the adjoint action, this amounts to showing that the following identities are satisfied for every $r \in \mathbb{Z}$:

$$\sum_{n,m\geqslant 0, n-m=r} \frac{(-x)^m q^{-n+m(m+1)/2} q^{mh}}{[m]![n]! \prod_{p=1}^{n}(xq^p - x^{-1}q^{-p})} \mathrm{e}^n f^m \tag{40}$$

$$= \sum_{k,l\geqslant 0, l-k=r} \frac{q^{\frac{k(k+1)}{2}}(-x)^k q^{kh}}{(q^2x^2; q^2)_{h+k}[k]!} \frac{(-x)^l}{(l)_{q^{-1}}!} f^k \mathrm{e}^l. \tag{41}$$

To prove this identity it is sufficient to show that it holds in any finite-dimensional representation of $U_q(sl(2))$. The simple $(p+1)$-dimensional module of $U_q(sl(2))$ has a basis v_k with $k = 0, \ldots, p$, with the following action:

$$hv_k = (p-2k)v_k \qquad ev_k = [p-k+1]_q v_{k-1} \qquad fv_k = [k+1]_q v_{k+1}. \tag{42}$$

We will show the proof for $r \geqslant 0$, the proof is similar for $r < 0$. A straightforward computation shows that the left-hand side (40) acting on v_k is equal to

$$\frac{[p-k+r]!q^{-r}}{[p-k]![r]!\prod_{p=1}^{r}(xq^{p}-x^{-1}q^{-p})}{}_2\varphi_2(q^{2(k+1)},q^{2(k-p)};q^{2(r+1)},x^2q^{2(r+1)};q^2)$$
$$\times(x^2q^{2(p-2k+2r+1)})v_{k-r} \tag{43}$$

whereas the action of the right-hand side (41) acting on v_k gives

$$\frac{[p-k+r]!(-x)^r q^{\frac{r(r-1)}{2}}}{[p-k]![r]!(q^2x^2;q^2)_{p-2k+2r}}{}_2\varphi_2(q^{2(p+r-k+1)},q^{2(r-k)};q^{2(r+1)},x^2q^{2(p-2k+2r+1)};q^2)$$
$$\times(x^2q^{2(r+1)})v_{k-r}. \tag{44}$$

These two expressions are equal thanks to the following identity on hypergeometric functions,

$${}_2\varphi_2(a,b;c,d;q)(cd/ab)=\frac{(cd/ab;q)_\infty}{(d;q)_\infty}{}_2\varphi_2(c/a,c/b;c,cd/ab;q)(d) \tag{45}$$

which is proved using the transformation of ${}_3\varphi_2$ series (formula III.9) of the appendix of the Gasper–Rahman book [15].

It remains to show that $\tilde{N}_-(x) = N_-(x)$, this is easily proved by equating the projections on the second space of the two expressions of $F(x)$ as a series (10) and as a product (14). □

3. Conclusion

We hope that this result will trigger some further work on this subject. Thirteen years after the work of Cremmer–Gervais, there is still no universal formula for $M(x)$ for $U_q(\mathfrak{g})$ with $\mathrm{rank}(\mathfrak{g}) > 1$. This problem has certainly a reasonable answer. From the study of the $U_q(sl(2))$ case it shows that the quantum Weyl group will certainly play a decisive role. There are questions that arose in the discussions with experts on the subject and which should be answered:

- What is the validity of the conjecture 1? What is the reason for the appearance of a specific Belavin–Drinfeld triple?
- Can it be generalized to other Kac–Moody algebras of finite type or affine type?
- In the affirmative case, can it be used to define an IRF-vertex transform in the affine case with potential applications to integrable systems?
- What is the relation, if any, between $M(x)$ and the dynamical quantum Weyl group [16]?

Added comment. The referee of this paper pointed out that H Rosengren has proved a factorization result of $M(x)$ in [17] in a different context.

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